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Infinite Player Noncooperative Games with Vector Payoffs Under Relative Pseudomonotonicity

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Abstract. In this paper we consider the Nash equilibrium problem for infinite player games with vector payoffs in a topological vector space setting. By employing new concepts of relative (pseudo)monotonicity, we establish several existence results of solutions for usual and normalized vector equilibria. The results strengthen existence results for vector equilibrium problems, which were based on classical pseudomonotonicity concepts. They also extend previous results for vector variational inequalities and finite player games under relative (pseudo)monotonicity.

Key words: existence results, infinite player games, Nash equilibria, relative pseudomonotonicity, vector payoffs

1. Introduction

Many equilibrium-type problems arising for example in economics, game theory and transportation can be formulated as the problem of finding an equilibrium point in a noncooperative game; see e.g. [11, 14, 15]. In addition to games with scalar payoffs their vector extensions have been studied extensively; see e.g. [3, 6, 8, 13]. Traditionally most existence results for scalar and vector equilibrium problems are based on fixed point techniques which require both the continuity and compactness (or proper coercivity) assumptions in the same topology. However these assumptions are too restrictive for applications in infinite-dimensional spaces. They can be relaxed by using (generalized) monotonicity properties of the so-called normalized cost bifunction; see e.g. [4, 6, 7].

In [9] Konnov introduced new generalized monotonicity concepts, which are based on the invariance of solution sets of decomposable equilibrium problems with respect to certain linear transformations. He proved existence and uniqueness results for scalar variational inequalities in a Banach space. These so-called relative monotonicity concepts can be regarded as intermediary between standard monotonicity and order monotonicity. In [1] the results in [9] were extended to vector variational inequalities. In a previous paper [2] the authors of the present study suggested an approach, which is based on these new (generalized) monotonicity concepts in order to prove the existence of an equilibrium point in a noncooperative game with vector payoffs and a finite number of players.

In this paper we consider a noncooperative game with vector payoffs and an infinite number of players in a topological vector space setting. We give some conditions under which a vector equilibrium problem (VEP) and a Nash vector equilibrium problem (VNEP) are equivalent. We establish several existence results for both problems and also present their specialization for games with scalar payoffs.

Let I be a countable set of indices. For each $i \in I$ let E_i be a real topological vector space. Let F be a real topological vector space with a partial order $\geqslant$ induced by a pointed closed convex and solid cone C. Thus for $y', y'' \in F, y' \geqslant y''$ is equivalent to $y' - y'' \in C, y' > y''$ is equivalent to $y' - y'' \in \operatorname{int} C$ and $y' \not> y''$ is equivalent to $y' - y'' \notin \operatorname{int} C$. We consider an infinite player noncooperative game where the ith player has a strategy set $X_i \subseteq E_i$ which is assumed to be nonempty, convex and closed and a utility function $f_i : E \to F$ with the joint strategy space

$$E = \prod_{i \in I} E_i.$$

For a point $x \in E$, we denote by x_{-i} its projection onto $\prod_{k \neq i} E_k$. Also we denote by $\mathbb{R}^I$ the set of all sequences with elements in $\mathbb{R}$, namely $\mathbb{R}^I = \{\mu \mid \mu_i \in \mathbb{R}, i \in I\}$, and the sets

$$\mathbb{R}_> = \{\mu \in \mathbb{R} \mid \mu > 0\},$$
$$\mathbb{R}^I_> = \{\mu \in \mathbb{R}^I \mid \mu_i > 0, i \in I\}$$

and

$$X = \prod_{i \in I} X_i. \tag{1}$$

Then the *vector Nash equilibrium problem* (VNEP) is to find an element $x^* = (x_i^*)_{i \in I} \in X$ such that

$$f_i(x^*_{-i}, y_i) \not> f_i(x^*), \quad \forall y_i \in X_i, \ \forall i \in I. \tag{2}$$

For each $i \in I$, we set

$$\varphi_i(x, y_i) = f_i(x) - f_i(x_{-i}, y_i).$$

Then VNEP (2) can be rewritten as follows: find an element $x^* = (x_i^*)_{i \in I} \in X$ such that

$$\varphi_i(x^*, y_i) \not< 0, \quad \forall y_i \in X_i, \ \forall i \in I. \tag{3}$$

Following [8,12], we consider the bifunction

$$\Phi(x, y) = \sum_{i \in I} \varphi_i(x, y_i) \tag{4}$$

defined on the set of pairs (x, y) for which the series on the right-hand side of (4) is unconditionally convergent.

Note that $\Phi(x, x) = 0$ for each $x \in X$. If $\operatorname{dom}\Phi = X \times X$, we can consider the following *vector equilibrium problem* (VEP) of finding an element $x^* = (x_i^*)_{i \in I} \in X$ such that

$$\Phi(x^*, y) \not< 0, \quad \forall y \in X. \tag{5}$$

Together with VEP (4), (5) we consider its dual formulation [10], which is to find an element $x^* = (x_i^*)_{i \in I} \in X$ such that

$$\Phi(y, x^*) \not> 0, \quad \forall y \in X. \tag{6}$$

We denote by X^N, X^* and X^d the set of solutions of problems (2) (or equivalently, (3)), (5) and (6), respectively.

2. Preliminary Results

We first recall some relationships between VEP and VNEP under continuity-type and monotonicity-type assumptions on Φ.

DEFINITION 1. A function $Q : X \to F$ is said to be

(a) *convex*, if for each pair of points $x \in X, y \in X$ and for all $\alpha \in [0, 1]$ we have

$$Q(\alpha x + (1 - \alpha) y) \leqslant \alpha Q(x) + (1 - \alpha) Q(y);$$

(b) *quasiconvex*, if for each pair of points $x \in X, y \in X$ and for all $\alpha \in [0, 1]$ we have either $Q(\alpha x + (1 - \alpha) y \leqslant Q(x)$ or $Q(\alpha x + (1 - \alpha) y) \leqslant Q(y)$;

(c) *explicitly quasiconvex*, if it is quasiconvex and for each pair of points $x \in X, y \in X$ such that $Q(x) < Q(y)$ and for all $\alpha \in (0, 1)$ we have

$$Q(\alpha x + (1-\alpha)y) < Q(y);$$

(d) *u-hemicontinuous*, if for each pair of points $x \in X, y \in X$ and for all $\alpha \in [0, 1]$ the mapping $\alpha \mapsto Q(\alpha x + (1-\alpha)y)$ is continuous at 0^+;
(e) *lower semicontinuous*, if for each $v \in F$ the level set $\{x \in X \mid Q(x) \not> v\}$ is closed.

The following inclusion is straightforward.

LEMMA 1. $X^* \subseteq X^N$.

The proof follows immediately from (3) to (5).

We intend to obtain the reverse inclusion and introduce an additional property.

DEFINITION 2. The bifunction $\Phi\colon X \times X \to F$ defined by (4) is *pseudo P-monotone* if for all $x, y \in X$, we have

$$\varphi_i(x, y_i) \not< 0 \ \forall i \in I \Longrightarrow \Phi(y, x) \not> 0.$$

LEMMA 2. *If the bifunction* $\Phi : X \times X \to F$ *defined by* (4) *is pseudo P-monotone, then* $X^N \subseteq X^d$.

The proof is straightforward.

LEMMA 3. *Suppose that* $\Phi(\cdot, y)$ *is u-hemicontinuous for each* $y \in X$ *and that* $\Phi(x, \cdot)$ *is explicitly quasiconvex for each* $x \in X$. *Then* $X^d \subseteq X^*$.

The proof follows from Proposition 3.1 in [4]. Explicit quasiconvexity of $\Phi(x, \cdot)$ can be relaxed to a certain type of generalized convexity as introduced in [4].

Combining Lemmas 1–3, we obtain the following equivalence result.

PROPOSITION 1. *Suppose that the bifunction* $\Phi : X \times X \to F$ *defined by* (4) *is pseudo P-monotone,* $\Phi(\cdot, y)$ *is u-hemicontinuous for each* $y \in X$ *and* $\Phi(x, \cdot)$ *is explicitly quasiconvex for each* $x \in X$. *Then VEP* (5) *is equivalent to VNEP* (3).

In order to establish existence results for VNEP and VEP we need the following well-known Ky Fan Lemma; e.g., see [5, Corollary 1].

PROPOSITION 2. *Let X and Y by nonempty sets in a topological vector space E and $Z: X \to 2^Y$ be such that*

(i) *for each $x \in X$, $Z(x)$ is closed in Y;*
(ii) *for each finite subset $\{x^1, \ldots, x^n\}$ of X, its convex hull is contained in the corresponding union $\bigcup_{i=1}^n Z(x^i)$;*
(iii) *there exists a point $\tilde{x} \in X$ such that $Z(\tilde{x})$ is compact.*
Then,

$$\bigcap_{x \in X} Z(x) \neq \emptyset.$$

3. Relative Monotonicity-Type Properties for Bifunctions

We begin our considerations with the definition of weight bifunctions.

DEFINITION 3. We say that $\gamma : X \times X \to \mathbb{R}^I_>$ is a *weight bifunctions* associated with VNEP (3) if γ is a family of bifunctions $(\gamma_i)_{i\in I}$ where $\gamma_i : X \times X_i \to \mathbb{R}_>$ for each $i \in I$.

Given a weight bifunction $\gamma : X \times X \to \mathbb{R}^I_>$ associated with VNEP (3), we can define the bifunction $[\gamma\Phi] : X \times X \to F$ as follows:

$$[\gamma\Phi](x, y) = \sum_{i\in I} \gamma_i(x, y_i)\varphi_i(x, y_i).$$

We can consider a primal–dual pair of VEP associated with the bifunction $[\gamma\Phi]$. The primal VEP is to find an element $x^* = (x_i^*)_{i\in I} \in X$ such that

$$[\gamma\Phi](x^*, y) \not< 0, \quad \forall y \in X \tag{7}$$

and the dual VEP is to find an element $x^* = (x_i^*)_{i\in I} \in X$ such that

$$[\gamma\Phi](y, x^*) \not> 0, \quad \forall y \in X. \tag{8}$$

We denote by X_γ^* and X_γ^d the sets of solutions of problems (7) and (8), respectively. We now adjust Lemmas 1 and 3 to these problems.

LEMMA 4. *Suppose that $\gamma: X \times X \to R^I_>$ is a weight bifunction associated with VNEP* (3). *It follows that*

(i) *$X_\gamma^* \subseteq X^N$,*
(ii) *if the bifunction $\Phi: X \times X \to F$ is defined by* (4), *$[\gamma\Phi](\cdot, y)$ is u-hemicontinuous for each $y \in X$ and $[\gamma\Phi](x, \cdot)$ is explicitly quasiconvex for each $x \in X$, then $X_\gamma^d \subseteq X_\gamma^*$.*

We now introduce new concepts of generalized monotonicity for bifunctions defined on infinite product sets.

DEFINITION 4. The bifunction $\Phi: X \times X \to F$ defined by (4) is said to be

(a) *relatively monotone* with respect to α, β (in short, (α, β)-*monotone*) if there exist weight bifunctions $\alpha, \beta : X \times X \to \mathbb{R}^I_>$ associated with VNEP (3) such that for all $x, y \in X$ we have

$$[\beta\Phi](x, y) + [\alpha\Phi](y, x) \leqslant 0$$

and $\mathrm{dom}[\alpha\Phi] = \mathrm{dom}[\beta\Phi] = X \times X$;

(b) *relatively pseudomonotone* with respect to α, β (in short, (α, β)-*pseudomonotone*) if there exist weight bifunctions $\alpha, \beta : X \times X \to \mathbb{R}^I_>$ associated with VNEP (3) such that for all $x, y \in X$ we have

$$[\beta\Phi](x, y) \not< 0 \Longrightarrow [\alpha\Phi](y, x) \not> 0$$

and $\mathrm{dom}[\alpha\Phi] = \mathrm{dom}[\beta\Phi] = X \times X$.

These concepts extend similar ones of generalized monotonicity in [2]. But in the present study they are defined for weight bifunctions instead of functions and are adjusted for infinite product sets. All of these extend classical (pseudo)monotonicity-type concepts [4, 7].

4. Existence Results

We begin with establishing an existence result for VNEP (3) on compact sets.

THEOREM 1. *Let X be nonempty, convex and compact. Suppose that the bifunction Φ defined by* (4) *is (α, β)-pseudomonotone. Suppose also that $[\alpha\Phi](\cdot, y)$ is u-hemicontinuous for each $y \in X$, $[\alpha\Phi](x, \cdot)$ is explicitly quasiconvex and lower semicontinuous for each $x \in X$, and $[\beta\Phi](x, \cdot)$ is quasiconvex for each $x \in X$.*

Then VNEP (3) *is solvable.*

Proof. Define set-valued mappings $A, B: X \to 2^X$ by

$$A(y) = \{x \in X \mid [\alpha\Phi](y, x) \not> 0\}$$

and

$$B(y) = \{x \in X \mid [\beta\Phi](x, y) \not< 0\}.$$

The proof is done in three steps; see [10].

(i) $\bigcap_{y\in X}\overline{B(y)}\neq\emptyset$. Let z be in the convex hull of any finite subset $\{y^1,\dots,y^n\}$ of X. Then $z=\sum_{j=1}^n \mu_j y^j$ for some $\mu_j\geqslant 0,\ j=1,\dots,n$ and $\sum_{j=1}^n \mu_j=1$. If $z\notin\cup_{j=1}^n B(y^j)$, then we have

$$[\beta\Phi](z,y^j)<0,\quad \forall j=1,\dots,n.$$

However by quasiconvexity of $[\beta\Phi](z,\cdot)$ we have

$$0=[\beta\Phi](z,z)=[\beta\Phi]\left(z,\sum_{j=1}^n \mu_j y^j\right)\leqslant \max_{j=1,\dots,n}\left\{[\beta\Phi](z,y^j)\right\}<0$$

a contradiction. Therefore $z\in\bigcup_{j=1}^n B(y^j)$, and we see that the mapping $y\mapsto\overline{B(y)}$ satisfies all the assumptions of Proposition 2. Hence we obtain

$$\bigcap_{y\in X}\overline{B(y)}\neq\emptyset.$$

(ii) $\bigcap_{y\in X}A(y)\neq\emptyset$. From (α,β)-pseudomonotonicity of Φ it follows that $B(y)\subseteq A(y)$. Since $[\alpha\Phi](y,\cdot)$ is lower semicontinuous, $A(y)$ is closed for each $y\in X$. Therefore $\overline{B(y)}\subseteq A(y)$ and (i) now implies (ii).

(iii) $X^N\neq\emptyset$. From (ii) it follows that $X_\alpha^d\neq\emptyset$. Applying now Lemma 4 yields $X^N\neq\emptyset$, as desired

Thus VNEP (3) (or equivalently (2)) is solvable. □

Comparing the assumptions of Theorem 1 with those of corresponding results for finitely many players (see [2, Theorem 4.1]), we notice that explicit quasiconvexity of $[\alpha\Phi](x,\cdot)$ was added. On the other hand $\Phi(x,\cdot)$ may not possess generalized convexity-type properties in this theorem.

By employing the corresponding coercivity condition we obtain an existence result on noncompact sets.

COROLLARY 1. *Let X be nonempty, convex and closed. Suppose that the bifunction Φ defined by* (4) *is (α,β)-pseudomonotone. Suppose also that $[\alpha\Phi](\cdot,y)$ is u-hemicontinuous for each $y\in X$, $[\alpha\Phi](x,\cdot)$ is explicitly quasiconvex and lower semicontinuous for each $x\in X$, and $[\beta\Phi](x,\cdot)$ is quasiconvex for each $x\in X$. Furthermore assume that there exist a compact subset Y of E and a point $\tilde{y}\in Y\cap X$ such that*

$$\textit{either } [\beta\Phi](x,\tilde{y})<0,\ \forall x\in X\backslash Y \quad \text{or } [\alpha\Phi](\tilde{y},x)>0,\ \forall x\in X\backslash Y. \tag{9}$$

Then VNEP (3) *is solvable.*

Proof. In this case it suffices to follow the proof of Theorem 1 and observe that either $\overline{B(\tilde{y})} \subseteq Y$ or $A(\tilde{y}) \subseteq Y$ under the above assumptions. In fact, it follows that $\overline{B(\tilde{y})}$ is compact, hence the assertion of Step (i) is true due to Proposition 2 as well. □

Combining the above results with Proposition 1 we obtain an existence result for VEP (4), (5).

THEOREM 2. *Let X be nonempty and convex, and let the bifunction Φ defined by* (4) *satisfy the following conditions*:

(i) Φ *is pseudo P-monotone and* (α, β) - *pseudomonotone*;
(ii) $\Phi(\cdot, y)$ *and* $[\alpha\Phi](\cdot, y)$ *are u-hemicontinuous for each* $y \in X$;
(iii) $\Phi(x, \cdot)$ *is explicitly quasiconvex,* $[\beta\Phi](x, \cdot)$ *is quasiconvex and* $[\alpha\Phi]$ $(x, \cdot)$ *is explicitly quasi-convex and lower semicontinuous for each* $x \in X$.

Furthermore suppose that either X is compact or X is closed and there exist a compact subset Y of E and a point $\tilde{y} \in Y \cap X$ such that (9) *holds.*
Then VEP (5) *is solvable.*

5. Applications to Scalar EP

In this section we present specializations of the previous existence results and some uniqueness results for the scalar case. Throughout this section we assume that $F = \mathbb{R}$ and $C = \mathbb{R}_+$, the set of nonnegative numbers. Then the scalar Nash equilibrium problem consists in finding an element $x^* = (x_i^*)_{i \in I} \in X$ such that

$$f_i(x^*_{-i}, y_i) \leqslant f_i(x^*), \quad \forall y_i \in X_i, \ \forall i \in I \tag{10}$$

or equivalently,

$$\varphi_i(x^*, y_i) \geqslant 0, \quad \forall y_i \in X_i, \ \forall i \in I, \tag{11}$$

where

$$\varphi_i(x, y_i) = f_i(x) - f_i(x_{-i}, y_i), \quad \forall i \in I.$$

Problems (10) and (11) are scalar analogues of (2) and (3). Using (4) we can define the normalized equilibrium problem: find an element $x^* = (x_i^*)_{i \in I} \in X$ such that

$$\Phi(x^*, y) \geqslant 0, \quad \forall y \in X. \tag{12}$$

The corresponding dual problem is: find an element $x^*=(x_i^*)_{i\in I}\in X$ such that

$$\Phi(y,x^*)\leqslant 0, \quad \forall y\in X. \tag{13}$$

Evidently problems (12) and (13) are scalar analogues of (5) and (6). In the scalar case problem (12) also implies (11). The reverse implication holds if $\operatorname{dom}\Phi=X\times X$; i.e., Φ need not be pseudo P-monotone in general. In such a way we can also specialize the parametric problems (7) and (8) and the concepts of relative (pseudo) monotonicity.

From Theorems 1 and 2 and the above remarks we obtain the following existence result.

THEOREM 3. *Let X be nonempty and convex, and let the bifunction $\Phi\colon X\times X\to\mathbb{R}$ defined by* (4) *satisfy the following conditions:*

(i) Φ *is* (α,β)*-pseudomonotone*;
(ii) $[\alpha\Phi](\cdot,y)$ *is u-hemicontinuous for each* $y\in X$;
(iii) $[\beta\Phi](x,\cdot)$ *is quasiconvex and* $[\alpha\Phi](x,\cdot)$ *is explicitly quasiconvex and lower semicontinuous for each* $x\in X$;
(iv) *either X is compact or X is closed and there exist a compact subset Y of E and a point $\tilde{y}\in Y\cap X$ such that*

$$\textit{either } [\beta\Phi](x,\tilde{y})<0, \quad \forall x\in X\backslash Y \quad \text{or } [\alpha\Phi](\tilde{y},x)>0,\ \forall x\in X\backslash Y. \tag{14}$$

Then problem (10) *is solvable. If in addition* $\operatorname{dom}\Phi=X\times X$, *then problem* (12) *is solvable.*

We now introduce somewhat strengthened concepts of generalized monotonicity for scalar bifunctions on infinite product sets.

DEFINITION 5. The bifunction $\Phi:X\times X\to\mathbb{R}$ defined by (4) is said to be

(a) *strictly* (α,β)*-monotone* if there exist weight bifunctions $\alpha,\beta:X\times X\to\mathbb{R}^I_>$ associated with problem (11) such that for all $x,y\in X, x\neq y$ we have

$$[\beta\Phi](x,y)+[\alpha\Phi](y,x)<0$$

and $\operatorname{dom}[\alpha\Phi]=\operatorname{dom}[\beta\Phi]=X\times X$;

(b) *strictly*(α, β)*-pseudomonotone* if there exist weight bifunctions $\alpha, \beta : X \times X \to \mathbb{R}^l_>$ associated with problem (11) such that for all $x, y \in X, x \neq y$ we have

$$[\beta\Phi](x, y) \geqslant 0 \Longrightarrow [\alpha\Phi](y, x) < 0$$

and $\mathrm{dom}[\alpha\Phi] = \mathrm{dom}[\beta\Phi] = X \times X$.

These concepts extend similar concepts in [9] introduced for single-valued mappings. It is clear that each strictly (α, β)-monotone bifunction is strictly (α, β)-pseudomonotone, but the reverse is not true in general. We will utilize these concepts for establishing uniqueness results.

PROPOSITION 3. *Suppose that the bifunction* $\Phi: X \times X \to \mathbb{R}$ *defined by* (4) *is strictly* (α, β)*-pseudomonotone. Then problem* (10) *has at most one solution. If in addition* $\mathrm{dom}\,\Phi = X \times X$*, then the same is true for problem* (12).

Proof. Assume to the contrary that there exist at least two different solutions x' and x'' of problem (10). Then we have

$$[\beta\Phi](x', x'') \geqslant 0 \quad \text{and} \quad [\alpha\Phi](x'', x') \geqslant 0$$

and using the strict (α, β)-pseudomonotonicity now yields

$$[\alpha\Phi](x'', x') < 0,$$

a contradiction. □

Combining Theorem 3 and Proposition 3 we obtain the following existence and uniqueness result.

COROLLARY 2. *Suppose that all the assumptions of Theorem* 3 *are fulfilled with the exception of* (*i*) *which is replaced by the following*:

(i') Φ *is strictly* (α, β)*-pseudomonotone.*

Then problem (10) *has a unique solution. If in addition* $\mathrm{dom}\ \Phi = X \times X$, *then the same is true for problem* (12).

In a reflexive Banach space setting we can utilize some other coercivity condition instead of that in (14).

DEFINITION 6. Suppose that E is a reflexive Banach space and X is unbounded. The bifunction $\Phi : X \times X \to \mathbb{R}$ defined by (4) is said to be γ*-coercive* if there exists a weight bifunction $\gamma : X \times X \to \mathbb{R}^l_>$ associated with problem (11) such that

$$[\gamma\Phi](x, \tilde{y}) \to -\infty \text{ as } \|x\| \to \infty, \quad x \in X$$

for some $\tilde{y} \in X$.

THEOREM 4. *Let X be a nonempty and convex subset of a reflexive Banach space E. Suppose that the bifunction $\Phi : X \times X \to \mathbb{R}$ defined by* (4) *satisfies the following conditions*:

(i) Φ *is* (α, β)- *pseudomonotone*;
(ii) $[\alpha\Phi](\cdot, y)$ *is u-hemicontinuous for each* $y \in X$;
(iii) $[\beta\Phi](x, \cdot)$ *is quasiconvex and* $[\alpha\Phi](x, \cdot)$*is explicitly quasiconvex and lower semicontinuous for each* $x \in X$;
(iv) *either X is compact or X is closed and Φ is α-coercive.*

Then problem (10) *is solvable. If in addition* dom $\Phi = X \times X$, *then problem* (12) *is solvable.*

Proof. It is sufficient to prove that there exists a solution of problem (10) under the α-coercivity condition. Let B_r denote the closed ball (under the norm) of E with center at 0 and radius r. Then for each $r > 0$ there exists a solution $x^r \in X_r = X \cap B_r$ for the following problem:

$$[\alpha\Phi](x^r, y) \geqslant 0 \quad \forall y \in X_r$$

due to Theorem 3. We observe that the set $\{x^r | r > 0\}$ must be bounded since otherwise we could choose r large enough such that the α-coercivity of Φ would yield

$$[\alpha\Phi](x^r, \tilde{y}) < 0,$$

a contradiction. Therefore there exists r such that $\|x^r\| < r$. Now for each $y \in X$ we can choose $\varepsilon > 0$ small enough such that $x^r + \varepsilon(y - x^r) \in X_r$. Then by explicit quasiconvexity of $[\alpha\Phi](x^r, \cdot)$ we have

$$0 \leqslant [\alpha\Phi](x^r, x^r + \varepsilon(y - x^r)) < \max\{[\alpha\Phi](x^r, y), 0\} = 0$$

if $[\alpha\Phi](x^r, y) < 0$, a contradiction. Thus

$$[\alpha\Phi](x^r, y) \geqslant 0 \quad \forall y \in X.$$

It follows that x_r is a solution of problem (10), and the result follows. □

Combining Theorem 4 and Proposition 3 yields another existence and uniqueness result.

COROLLARY 3. *Suppose that all the assumptions of Theorem* 4 *are fulfilled with the exception of* (*i*), *which is replaced with* (i') *of Corollary* 2. *Then problem* (10) *has a unique solution. If in addition* dom $\Phi = X \times X$, *then the same is true for problem* (12).

6. Scalarization of VEP

In this section we introduce some other relative (pseudo)monotonicity concepts and present existence results for VEP which are based on solving an appropriately scalarized version of the problem.

Given an element $z \in F^*$, the dual space of F, and a bifunction $\Phi : X \times X \to F$ defined by (4), we introduce the bifunction $\Phi_z : X \times X \to \mathbb{R}$ by

$$\Phi_z(x, y) = (z, \Phi(x, y))$$

for $x, y \in X$. Similarly for a weight bifunction $\alpha : X \times X \to \mathbb{R}^I_>$ associated with VNEP (3) we define the weighted bifunction $[\alpha\Phi_z] : X \times X \to \mathbb{R}$ by

$$[\alpha\Phi_z](x, y) = (z, [\alpha\Phi](x, y))$$

for $x, y \in X$. We set

$$H_z = \{f \in F | (z, f) \geqslant 0\}.$$

Now we introduce relative (pseudo)monotonicity concepts which are different from those in Definition 4.

DEFINITION 7. Let z be an element in $F^* \backslash \{0\}$. The bifunction $\Phi : X \times X \to F$ defined by (4) is said to be

(a) (α, β)-*monotone* with respect to z if there exist weight bifunctions $\alpha, \beta : X \times X \to \mathbb{R}^I_>$ associated with VNEP (3) such that for all $x, y \in X$ we have

$$[\beta\Phi](x, y) + [\alpha\Phi](y, x) \in -H_z$$

and dom $[\alpha\Phi] =$ dom $[\beta\Phi] = X \times X$;

(b) (α, β)-*pseudomonotone* with respect to z if there exist weight bifunctions $\alpha, \beta : X \times X \to \mathbb{R}^I_>$ associated with VNEP (3) such that for all $x, y \in X$ we have

$$[\beta\Phi](x, y) \in H_z \Longrightarrow [\alpha\Phi](y, x) \in -H_z$$

and dom$[\alpha\Phi] =$ dom$[\beta\Phi] = X \times X$.

It is clear that (α, β)-monotonicity with respect to z implies (α, β)-pseudomonotonicity with respect to z. But the reverse is not true in general. Also if $C \subseteq H_z$, then relative monotonicity with respect to (α, β) implies (α, β)-monotonicity with respect to z.

PROPOSITION 4. *The bifunction* $\Phi : X \times X \to F$ *defined by* (4) *is* (α, β)-*monotone* (*respectively,* (α, β)-*pseudomonotone*) *with respect to* z *for some* z *in* $F^*\backslash\{0\}$ *if and only if the bifunction* $\Phi_z : X \times X \to \mathbb{R}$ *is* (α, β)-*monotone* (*respectively,* (α, β)-*pseudomonotone*).

Proof. Since for all $x, y \in X$ the relation

$$[\alpha\Phi_z](y, x) + [\beta\Phi_z(x, y)] \leqslant 0$$

is equivalent to

$$[\alpha\Phi](y, x) + [\beta\Phi](x, y) \in -H_z,$$

we see that the assertion is true for (α, β)-monotonicity. Suppose now that Φ is (α, β)-pseudomonotone with respect to z and that

$$(z, [\beta\Phi](x, y)) = [\beta\Phi_z](x, y) \geqslant 0$$

for some $x, y \in X$. It follows that $[\beta\Phi](x, y) \in H_z$ and $[\alpha\Phi](y, x) \in -H_z$. Hence $[\alpha\Phi_z](y, x) \leqslant 0$, i.e., Φ_z is (α, β)-pseudomonotone. Conversely, if Φ_z is (α, β)-pseudomonotone and $[\beta\Phi](x, y) \in H_z$ for some $x, y \in X$, then $[\beta\Phi_z](x, y) \geqslant 0$, hence $[\alpha\Phi_z](y, x) \leqslant 0$. Therefore $[\alpha\Phi](y, x) \in -H_z$, i.e., Φ is (α, β)-pseudomonotone with respect to z. Hence the assertion is true for (α, β)-pseudomonotonicity. □

In what follows we denote by C^* the dual cone of C, i.e.,

$$C^* = \{z \in F^* \mid (z, f) \geqslant 0 \quad \forall f \in C\}.$$

Combining the above property with Theorem 3 we obtain another existence result for VEP.

THEOREM 5. *Let* X *be nonempty and convex and let the bifunction* $\Phi :$ $X \times X \to F$ *defined by* (4) *satisfy the following conditions*:

(i) Φ *is* (α, β)-*pseudomonotone with respect to some* z *in* $C^*\backslash\{0\}$;
(ii) $[\alpha\Phi_z](\cdot, y)$ *is u-hemicontinuous for each* $y \in X$;
(iii) $[\beta\Phi_z](x, \cdot)$ *is quasiconvex and* $[\alpha\Phi_z](x, \cdot)$*is explicitly quasiconvex and lower semicontinuous for each* $x \in X$;

(iv) *either* X *is compact or* X *is closed and there exist a compact subset* Y *of* E *and a point* $\tilde{y} \in Y \cap X$ *such that*

$$\text{either } [\beta\Phi_z](x, \tilde{y}) < 0, \ \forall x \in X \backslash Y \quad \text{or } [\alpha\Phi_z](\tilde{y}, x) > 0, \ \forall x \in X \backslash Y.$$

Then VNEP (3) *is solvable. If in addition* $\mathrm{dom}\Phi = X \times X$, *then VEP* (5) *is solvable.*

Proof. Since $z \in C^* \backslash \{0\}$, the bifunction Φ_z is (α, β)-pseudomonotone due to Proposition 4. Besides, under the assumptions of the present theorem, it is easy to verify that the bifunction Φ_z satisfies all the assumptions of Theorem 3. Therefore there exists a point $x^* \in X$ such that

$$(z, \varphi_i(x^*, y_i)) \geqslant 0 \quad \forall y_i \in X_i, \ \forall i \in I. \tag{15}$$

This means that x^* solves VNEP (3). In fact, since $z \in C^* \backslash \{0\}$, $-\mathrm{int}H_z \supseteq -\mathrm{int}C$, so that $(z, \varphi_i(x^*, y_i)) \geqslant 0$ implies $\varphi_i(x^*, y_i) \in H_z$ and $\varphi_i(x^*, y_i) \notin -\mathrm{int}C$. In case $\mathrm{dom}\,\Phi = X \times X$, the value of $\Phi(x^*, y)$ exists. Moreover (15) yields

$$(z, \Phi(x^*, y)) \geqslant 0, \quad \forall y \in X.$$

Using the same argument we conclude that

$$\Phi(x^*, y) \not< 0,$$

i.e., x^* solves VEP (5). □

Observe that according to Theorem 5 the existence of a solution of the normalized VEP (5) can be established without assuming pseudo P-monotonicity of Φ, unlike in the result in Theorem 2.

Using a coercivity condition we obtain an additional existence result on noncompact sets in a reflexive Banach space setting.

DEFINITION 8. Suppose E is a reflexive Banach space. The bifunction $\Phi : X \times X \to F$ defined by (4) is said to be γ-*coercive* with respect to $z \in F^* \backslash \{0\}$ if the scalarized bifunction $\Phi_z : X \times X \to \mathbb{R}$ is γ-coercive.

The corresponding existence result is based on combining Proposition 4 and Theorem 4.

THEOREM 6. *Let* X *be a nonempty, convex, and closed subset of a reflexive Banach space* E. *Suppose that the bifunction* $\Phi : X \times X \to F$ *defined by* (4) *satisfies the following conditions*:

(i) Φ *is both* (α, β)-*pseudomonotone and* α-*coercive with respect to* $z \in C^* \backslash \{0\}$;

(ii) $[\alpha\Phi_z](\cdot, y)$ *is u-hemicontinuous for each* $y \in X$;

(iii) $[\beta\Phi_z](x, \cdot)$ *is quasiconvex and* $[\alpha\Phi_z](x, \cdot)$*is explicitly quasiconvex and lower semicontinuous for each* $x \in X$.

Then VNEP (3) *is solvable. If in addition* $\operatorname{dom}\Phi = X \times X$, *then VEP* (5) *is solvable.*

Proof. Since $z \in C^*\backslash\{0\}$, the bifunction Φ_z is (α, β)-pseudomonotone. Applying Theorem 4 now yields (15). Using the same argument as that in the proof of Theorem 5 we conclude that the assertions of the present theorem are true. □

7. Examples

In this section we present some examples which illustrate properties of the new concepts and their relationships with know ones. For simplicity we restrict ourselves to scalar utility functions.

EXAMPLE 1. We first consider the noncooperative infinite person game with the utility functions

$$f_i(x) = \sum_{j\in I, j\neq i} x_i^T A_{(i,j)} x_j,$$

where $x_i \in \mathbb{R}^{n_i}, x_j \in \mathbb{R}^{n_j}$ and the box-constrained strategy sets

$$X_i = \{z \in \mathbb{R}^{n_i} \mid \gamma_j^{(i)} \leqslant z_j \leqslant \delta_j^{(i)}, \quad j \in 1, \dots, n_i\}$$

for $i \in I$. Hence each $A_{(i,j)}$ is an $n_i \times n_j$ matrix. Set $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$. Then we have

$$\Phi(x, y) = \sum_{i\in I} (f_i(x) - f_i(x_{-i}, y_i)) = \sum_{i\in I} \sum_{j\neq i} \langle A_{(i,j)} x_j, x_i - y_i\rangle. \tag{16}$$

It follows that

$$\begin{aligned}[\alpha\Phi(x, y)] + [\alpha\Phi(y, x)] = &\sum_{i\in I} \sum_{j\neq i} \langle \alpha_i A_{(i,j)} x_j, x_i - y_i\rangle \\ &+ \sum_{i\in I} \sum_{j\neq i} \alpha_i \langle A_{(i,j)} y_j, y_i - x_i\rangle.\end{aligned}$$

If we suppose that $A_{(1,j)} = -\lambda A_{(j,1)}^T$ where $\lambda > 0$ is an arbitrary number and $A_{(i,j)} = -A_{(j,i)}^T$ for $i \neq 1$ and for all $j \in I$, then by setting $\alpha_1 = 1/\lambda$ and $\alpha_i = 1$ for $i > 1$ we have for each pair of points x, y

$$[\alpha\Phi(x, y)] + [\alpha\Phi(y, x)] = 0.$$

i.e., Φ is (α, α)-monotone. On the other hand, Φ may not even be pseudomonotone. For this, consider the simplest case where $n_1 = n_2 = 2$, $A_{(1,2)} = \left[\begin{smallmatrix} -1 & -2 \\ 0 & -2 \end{smallmatrix}\right]$, $A_{(2,1)} = -2A_{(1,2)}$ and $A_{(i,j)} = 0_{n_i \times n_j}$ for all $(n_i, n_j) \neq (1,2), (n_i, n_j) \neq (2,1)$. Setting $x_1 = [1.1\ \ 1]^T, x_2 = [3\ \ 3]^T, y_1 = [2\ \ 2]^T, y_2 = [4\ \ 4]^T$ we have

$$\Phi(x, y) = 3.5 > 0 \quad \text{and} \quad \Phi(y, x) = 1.2 > 0,$$

i.e., Φ is not pseudomonotone.

If we consider the case of $n_1 = n_2 = 2$, $A_{(1,2)} = \left[\begin{smallmatrix} -1 & 0 \\ 0 & -2 \end{smallmatrix}\right]$, $A_{(2,1)} = -2A_{(1,2)}$ and $A_{(i,j)} = 0_{n_i \times n_j}$ for all $(n_i, n_j) \neq (1,2), (n_i, n_j) \neq (2,1)$, then Φ is not pseudomonotone either. In fact for $x_1 = [1\ \ 1.1]^T, x_2 = [0\ \ 3]^T, y_1 = [2\ \ 2]^T, y_2 = [0\ \ 4]^T$ we have

$$\Phi(x, y) = 1.0 > 0 \quad \text{and} \quad \Phi(y, x) = 0.8 > 0.$$

It is also true that Φ is pseudo P-monotone. In fact considering the box-constrained set X with $x_3 = 0$, we obtain $\varphi_1(x, y_1) = \varphi_2(x, y_2) = 0$ as $\Phi(y, x) = 0$. Other assumptions of Theorem 2 are also satisfied.

Next, we can add nonlinear terms to the previous functions as follows:

$$f_i(x) = \sum_{j \in I, j \neq i} x_i^T A_{(i,j)} x_j + \psi_i(x_i),$$

where $\psi_i : X_i \to \mathbb{R}$ is concave and upper semicontinuous for $i \in I$. Then (16) is still true, i.e., Φ is also (α, α)-monotone. At the same time, we cannot apply the techniques based on classical pseudomonotonicity properties.

We now turn to the case of variable weight bifunctions.

EXAMPLE 2. We consider the variational inequality: find $x \in X$ such that

$$\sum_{i \in I} G_i(x)(y_i - x_i) \geqslant 0 \quad \forall y \in X,$$

where $I = \{1, \ldots, n\}$, $G_i : \mathbb{R}_+ \to \mathbb{R}$ is of the form

$$G_i(x) = [\langle a_i, x \rangle + \delta_i]^{-1} F_i(x_i) + \gamma_i,$$

$a_i \in \mathbb{R}^I_+, \gamma_i \geqslant 0, \delta_i > 0$ for $i \in I$, and $F : \mathbb{R}^I_+ \to \mathbb{R}^I$ is a monotone continuous mapping. Here we set

$$\mathbb{R}_+ = \{\mu \in \mathbb{R} \mid \mu \geqslant 0\}$$

and

$$\mathbb{R}^I_+ = \{\mu \in \mathbb{R}^I \mid \mu_i \geqslant 0, i \in I\}.$$

Choose X to be the box-constrained set in (1). Then the above variational inequality becomes a particular case of the scalar Nash equilibrium problem (11) where

$$\varphi(x, y_i) = G_i(x)(y_i - x_i),$$

which is equivalent to problem (12) with the cost bifunction $\Phi(x, y)$ defined in (4). Clearly this bifunction is not pseudomonotone in general. At the same time we can define the weight bifunctions $\alpha_i : \mathbb{R}_+^I \times \mathbb{R}_+ \to \mathbb{R}_+$ as follows:

$$\alpha_i(x, y_i) = \langle a_i, x\rangle + \delta_i.$$

Then for each pair of points $x, y \in \mathbb{R}_+^I$ we have

$$\begin{aligned}
&[\alpha\Phi](x, y) + [\alpha\Phi](y, x) \\
&\quad = \sum_{i \in I} [F_i(x) - F_i(y)](y_i - x_i) + \sum_{i \in I} \gamma_i \langle (y_i - x_i) a_i, x - y\rangle \\
&\quad \leqslant (y - x)^T (\gamma A)(x - y) \leqslant 0
\end{aligned}$$

if the matrix (γA) with rows $\gamma_i a_i, i \in I$, is positive semidefinite. It means that Φ is (α, α)-monotone. Moreover this property remains true if we assume I to be countable. In both cases we can apply the results of the previous sections to investigate existence and uniqueness properties.

Thus even these rather simple examples illustrate the fact that the new concepts extend the usual generalized monotonicity properties significantly. These new concepts deserve further investigation. They will be studied in a forthcoming paper.

References

1. Allevi, E., Gnudi, A. and Konnov, I.V. (2001), Generalized vector variational inequalities over product sets, *Nonlinear Analysis* 47, 573–582.
2. Allevi, E., Gnudi, A., Konnov, I.V. and Schaible, S. (2003), Noncooperative games with vector payoff under relative pseudomonotonicity, *Journal of Optimization Theory and Applications* 118, 245–254.
3. Blackwell, D. (1956), An analogue of the minimax theorem for vector payoffs, *Pacific Journal of Mathematics* 6, 1–8.
4. Bianchi, M., Hadjisavvas, N. and Schaible, S. (1997), Vector equilibrium problems with generalized monotone bifunctions, *Journal of Optimization Theory and Applications* 92, 527–542.
5. Fan, K. (1984), Some properties of convex sets related to fixed-point theorems, *Mathematische Annalen* 266, 519–537.
6. Giannessi, F. (2000), *Vector Variational Inequslities and Vector Equilibria. Mathematical Theories*, Kluwer Academic Publishers, Dordrecht, The Netherlands.

7. Hadjisavvas, N. and Schaible, S. (1998), Quasimonotonicity and pseudomonotonicity in variational inequalities and equilibrium problems. In: Crouzeix, J.-P., Martinez-Legaz, J.E. and Volle, M. (eds.), *Generalized Convexity, Generalized Monotonicity*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 257–275.
8. Konnov, I.V. (1995), Combined relaxation methods for solving vector equilibrium problems, *Russ. Mathematics* (Iz. VUZ) 39, (12), 51–59.
9. Konnov, I.V. (2001), Relatively monotone variational inequalities over product sets, *Operations Research Letters* 28, 21–26.
10. Konnov, I.V. and Schaible, S. (2000), Duality for equilibrium problems under generalized monotonicity, *Journal of Optimization Theory and Applications* 104, 395–408.
11. Nagurney, A. (1993), *Network Economics: A Variational Inequality Approach*, Kluwer Academic Publishers, Dordrecht, The Netherlands.
12. Nikaido, H. and Isoda, K. (1955), Note on noncooperative convex games, *Pacific Journal of Mathematics* 5, 807–815.
13. Podinovskii, V.V. and Nogin, V.D. (1982), *Pareto-optimal Solutions of Multicriteria Problems*, Nauka, Moscow. (in Russian).
14. Rosen, J.B. (1965), Existence and uniqueness of equilibrium points for concave n-person games, *Econometrica* 33, 520–534.
15. von Neumann, J. and Morgenstern, O. (1953), *Game Theory and Economic Behavior*, Princeton University Press, Princeton, N.J.